HIGH ORDER SCHEMES ON CRAZY MOVING VORONOI MESHES

ABSTRACT

The development of new, accurate, reliable, and efficient numerical methods for solving *hyperbolic* partial differential equations (PDE) has become a central discipline in present-day computational science. Indeed, the variety and broad applicability of hyperbolic equations cover a wide range of interesting phenomena, from human and hearth-science problems to the study of stars and galaxies, usually involving a *huge range* of space and time scales. Thus, a continuously increasing interest is focused on numerical strategies able to model such complex situations and to describe simultaneously multi-scale turbulent flow features, as well as (zero-scale) shocks and observer-size macrostructures, up to astronomical phenomena.

For this purpose, besides employing powerful software machinery and very high order accurate schemes, we also need to equip our numerical methods with *clever strategies* aimed at reducing particular sources of numerical errors. In particular, in order to reduce errors due to convective terms and to better track material interfaces and contact discontinuities, one can exploit the power of Lagrangian methods. However, ensuring the *high quality* of a moving mesh over long simulation times is difficult, therefore a certain degree of flexibility should be allowed in order to avoid mesh distortion, for example a slightly relaxed choice of the actual mesh velocity w.r.t the real fluid velocity, as well as the freedom of not only *moving* the control volumes, but really *evolving* their shapes and allowing topology and neighborhood changes.

With this in mind, we present here a *new* family of very *high order* accurate direct Arbitrary-Lagrangian-Eulerian (ALE) Finite Volume (FV) and Discontinuous Galerkin (DG) schemes for the solution of general nonlinear hyperbolic PDE systems on moving Voronoi meshes that are *regenerated* at each time step and which explicitly allow *topology changes* in time, in order to benefit simultaneously from high order methods, high quality grids and substantially reduced numerical dissipation. The *key ingredient* of our approach is the integration of a *space-time conservation formulation* of the governing PDE system over closed, non-overlapping *space-time* control volumes that are constructed from the moving, *regenerated* Voronoi meshes: this leads to also consider *crazy degenerate* control volumes that only exist in the space-time framework, and would not exist from a purely spatial point of view!

NUMERICAL METHOD

Introduction. In order to model a wide class of physical phenomena, we consider a very general formulation of the governing equations, namely all those which can be described by

$$\partial_t \mathbf{Q} + \nabla \cdot \mathbf{F}(\mathbf{Q}) + \mathbf{B}(\mathbf{Q}) \cdot \nabla \mathbf{Q} = \mathbf{S}(\mathbf{Q}), \quad \text{(PDE)}$$

where $\mathbf{Q}$ is the vector of the conserved variables, $\mathbf{F}$ the non linear flux, $\mathbf{B} \cdot \nabla \mathbf{Q}$ the non-conservative products, and $\mathbf{S}$ a nonlinear algebraic source term. Under this form, we can cast many physical models, from the simple shallow water system up to the Einstein field equations of general relativity. For example, in this work, we will present results for the Euler equations of gasdynamics with and

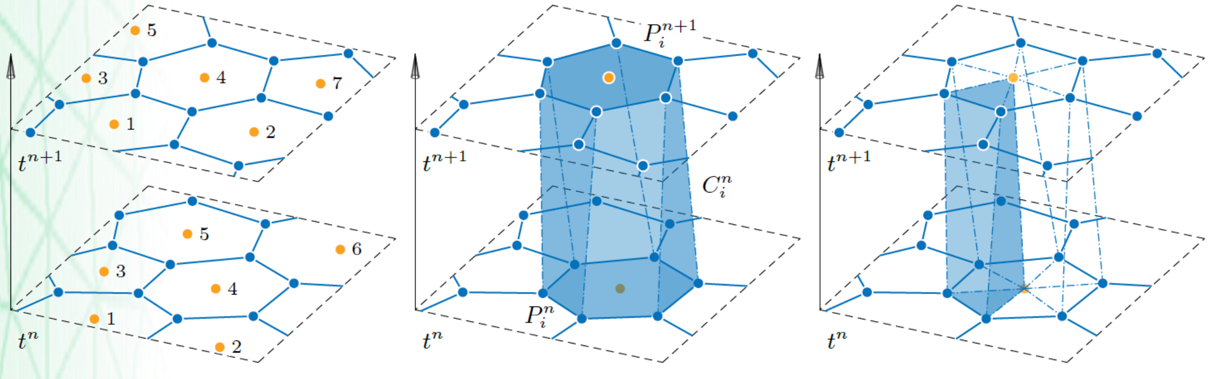

Figure 1: Space time connectivity without topology changes.

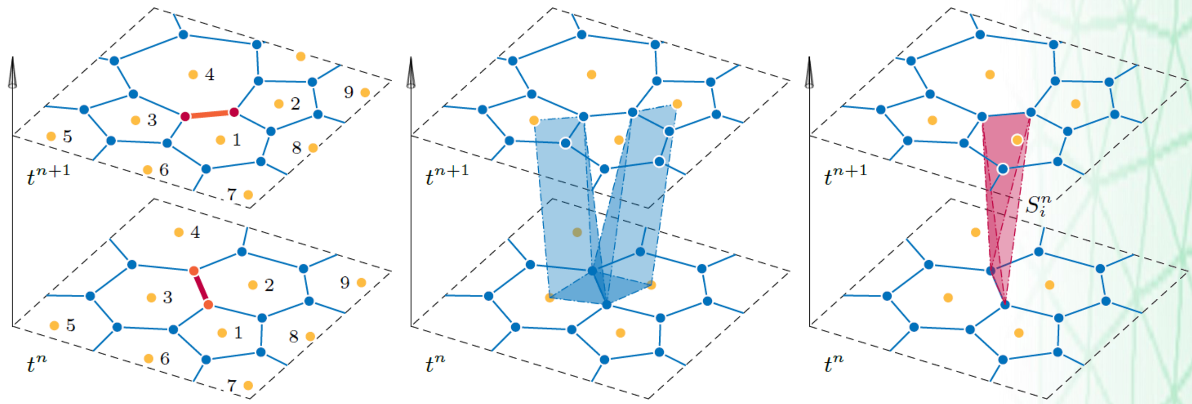

Figure 2: Space time connectivity with topology changes, degenerate sub-space-time control volumes and crazy sliver element.

without gravity, the magnetohydrodynamics (MHD) equations for plasma flows, and the GPR unified model of continuum mechanics.

At the beginning of the simulation, we discretize our moving domain by a centroid-based Voronoi-type tessellation built from a set of generators (the orange points in Figure 1), and we represent our data, the conserved variables $\mathbf{Q}$, via discontinuous high order polynomials in each Voronoi polygon. Then, we let the generators move with a velocity chosen as close as possible to the local fluid velocity, indeed computed mainly from a high order approximation of their pure Lagrangian trajectories, with small corrections obtained from a flow-adaptive mesh optimization technique. Thus, since the position of the generators is being updated at any time step, also the Voronoi tessellation may change at any time step. Then, a connection between two successive Voronoi tessellations has to be established in order to evolve the solution in time.

Direct ALE. The key idea of *direct* ALE methods (in contrast to *indirect* ones) consists in connecting two tessellations by means of so-called *space-time control volumes* C_i^n, and recover the unknown solution at the new time step $\mathbf{u}_h^{n+1}$ *directly* inside the new polygon P_i^{n+1}, from the data available at the previous time step $\mathbf{u}_h^n$ in P_i^n. This is achieved through the *integration*, over such control volumes, of the fluxes, the nonconservative products and the source terms, by means of a high order fully discrete predictor-corrector ADER method [1]. In this way, the need for any further remapping/remeshing steps is totally eliminated. By adopting the tilde symbol for referring to space-time quantities, our direct ALE scheme [2] reads

$$\int_{P_i^{n+1}} \tilde{\varphi}_k \mathbf{u}_h^{n+1} = \int_{P_i^n} \tilde{\varphi}_k \mathbf{u}_h^n$$
$$-\sum_j \int_{\partial C_{ij}^n} \tilde{\varphi}_k \mathcal{F}(\mathbf{q}_h^{n,-}, \mathbf{q}_h^{n,+}) \cdot \tilde{\mathbf{n}}$$
$$+\int_{C_i^n} \tilde{\nabla} \tilde{\varphi}_k \cdot \tilde{\mathbf{F}}(\mathbf{q}_h^n)$$
$$+\int_{C_i^n} \tilde{\varphi}_k \left(\mathbf{S}(\mathbf{q}_h^n) - \tilde{\mathbf{B}}(\mathbf{q}_h^n) \cdot \nabla \mathbf{q}_h^n \right),$$

Where $\tilde{\varphi}_k$ is a set of moving space-time basis functions, while $\mathbf{q}_h^{n,+}$ and $\mathbf{q}_h^{n,-}$ are high order space-time extrapolated data computed through the ADER predictor. Finally, $\mathcal{F}(\mathbf{q}_h^{n,-}, \mathbf{q}_h^{n,+})$ is an ALE numerical flux function which takes into account fluxes across space-time cell boundaries ∂C_{ij}^n as well as jump terms related to nonconservative products.

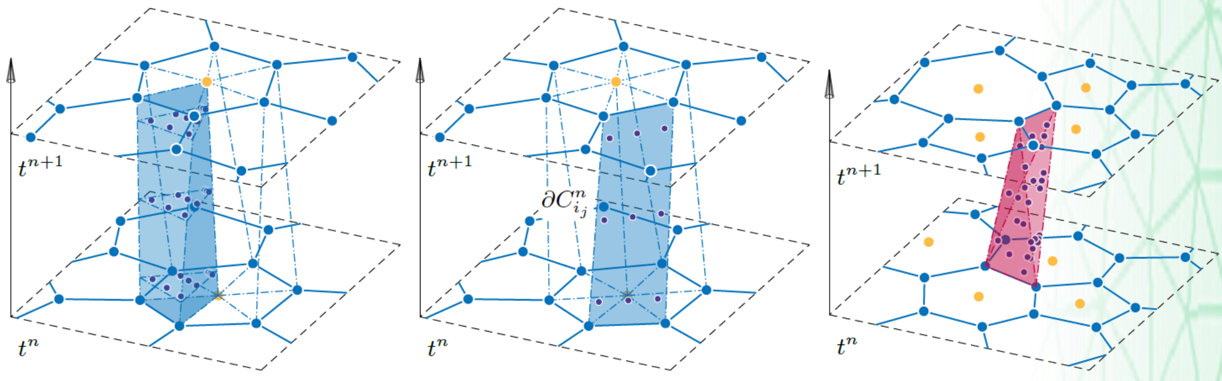

Figure 3: Space-time quadrature points for third order methods.

As evident from the numerical scheme, in order to compute the integrals with high order of accuracy, complete knowledge of the *space-time connectivity* between two consecutive time steps is required, as opposed to only the *spatial* information at the two time levels. When no topology changes occur, the space-time geometrical information is easily constructed by connecting via straight line segments the corresponding vertexes of each polygon, obtaining an oblique prism than can be further subdivided into a set of triangular oblique sub-prisms on which quadrature points are readily available (see Images 1 and 3).

Crazy elements. On the contrary, when a topology change occurs, as in Figure 2, i.e. the number of edges, the shape, and the neighbors of a polygon change between two consecutive time steps, the space-time connection between them induces the appearance of degenerate elements of two types: *(i).* degenerate *sub*-space-time control volumes, where either the top or bottom faces are degenerate triangles that are collapsed to a segment; *(ii).* and also *crazy sliver* space-time elements S_i^n. The first type of degenerate elements does not pose any problem, and was already treated in [3]. Instead, sliver elements are a completely new type of control volume. In particular, they do not exist neither at time t^n, nor at time t^{n+1}, since they coincide with an edge of the tessellation at the old and at the new time levels, and, as such, have zero area in space. However, they have a *non-negligible volume* in space-time. The difficulties related to this kind of elements are due to the fact that for them an initial condition is not clearly defined at time t^n, and that contributions across these elements should not be lost at time t^{n+1}, in order to ensure conservation. All the details on how to successfully extend our direct ALE also to *crazy* elements can be found in our recent paper [2].

We would like to emphasize that topology changes are fundamental for long time simulations in the ALE framework and our *crazy*, i.e. sliver, elements represent a *novel and robust* way to allow for a relatively simple space-time connection around a change of connectivity.

The predictor. The *predictor* step represents an essential ingredient for obtaining high order in time in a fully-discrete one-step procedure: it yields a *local* solution of the governing equations (PDE) *in the small* $\mathbf{q}_h^n$, inside each space-time element, including the crazy elements. The solution is *local* in the sense that it is obtained by only considering the initial data in each polygon, the governing equations and the geometry of C_i^n, without taking into account interactions between C_i^n and its neighbors. Such local solution is given for each standard space-time control volume C_i^n and each crazy control volume S_i^n, as a high order polynomial *in space and in time*, which serves as a predictor solution, to be used for evaluating all the integrals in the (ALE) *corrector* step, i.e. the final update of the solution from t^n to t^{n+1}.

A posteriori sub-cell FV limiter. High order schemes that can be seen as linear in the sense of Godunov [4], may develop spurious oscillations in presence of discontinuities. In order to prevent this phenomenon, in the case of a DG discretization we adopt an *a posteriori* limiting procedure based on the MOOD paradigm [5]: we first apply our unlimited ALE-DG scheme everywhere, and then (*a*

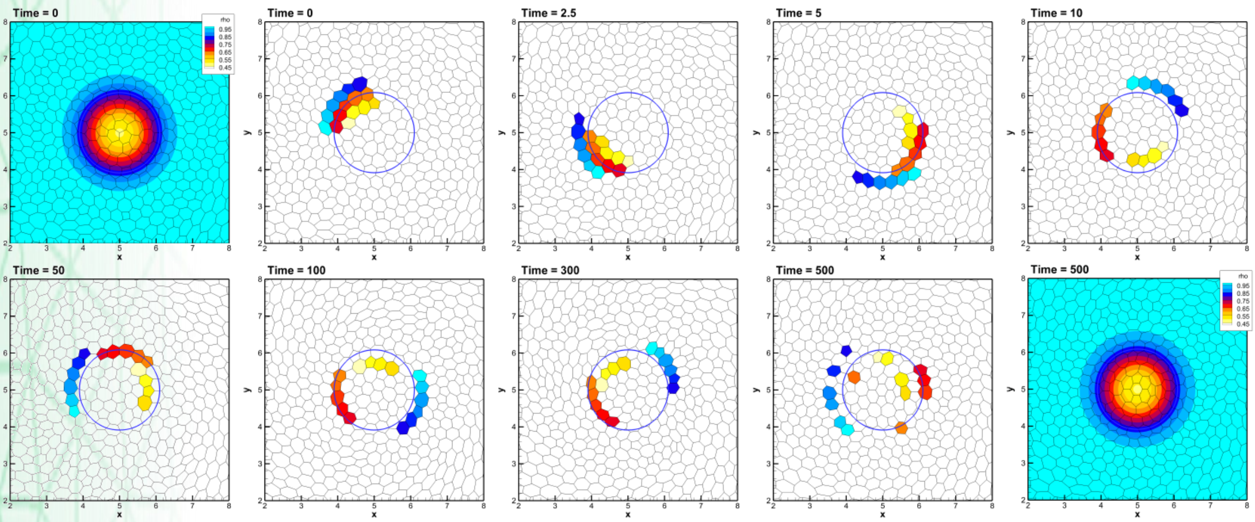

Figure 4: Stationary rotating vortex solved with our fourth order ALE-DG scheme. Density contours at t = 0 and t = 500 and position of a bunch of highlighted elements at different times. Note that the solution is well preserved for more than fifty complete loops and generator trajectories are perfectly circular!

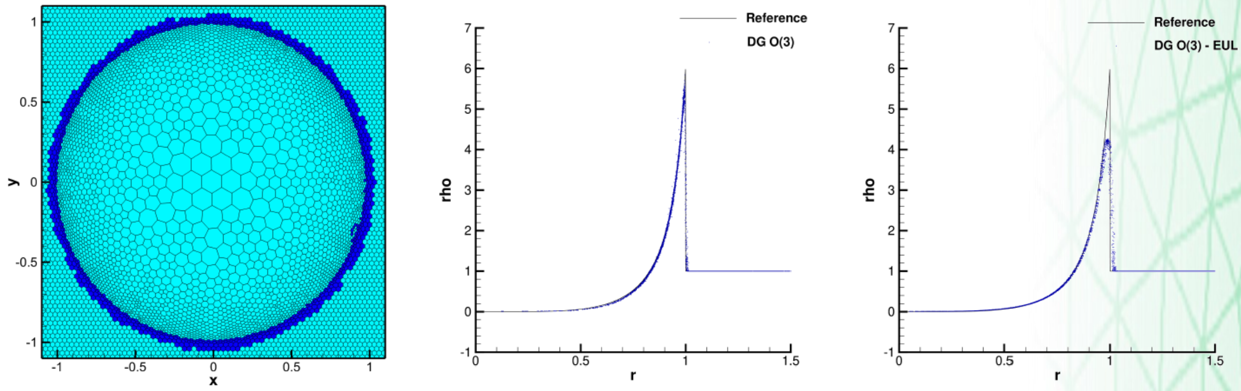

Figure 5: Sedov explosion problem solved with our third order ALE-DG scheme. The topology changes maintain a high quality mesh and the Lagrangian framework (middle) allows to obtain a scatter profile perfectly in agreement with the reference solution and much better than what obtainable with an Eulerian scheme (right) of the same order. Note that our a posteriori FV limiter activates exactly only at the shock location (blue cells in the left panel).

posteriori), at the end of each time step, we check the reliability of the obtained solution in each cell against physical and numerical admissibility criteria. Next, we mark as *troubled* those cells where the DG solution cannot be accepted. For the troubled cells we now repeat the time evolution by employing, instead of the DG scheme, a more robust FV method. Moreover, in order to maintain the accurate resolution of our original high order DG scheme, which would be lost when switching to a FV scheme, the FV scheme is applied on a *finer sub-cell grid* that accounts for recovering the optimal accuracy of the numerical method performing a reconstruction step.

Numerical Examples

We present here a selection of the variety of numerical tests available in [2], in order to show both the wide range of applicability of the proposed high order ALE scheme on moving Voronoi meshes with topology changes as well as its novelty with respect to the state-of-the-art. For *all* the presented test cases we have numerically verified that mass and volume conservation is respected up to *machine precision* at *any* time step, and that the same holds true for the Geometric Conservation Law (GCL) condition on *each* element, even when topology changes occur. Also, the order of convergence has been checked up to order five.

We want to focus in particular on vortex flows, see Figure 4, that give clear evidence of the advantages conveyed by the proposed algorithm. The correct density profile and a high quality mesh are conserved for times that are two orders of magnitude larger with respect to standard conforming ALE schemes, where mesh tangling would occur and stop the simulation much earlier. The position of a bunch of highlighted elements is shown at different times to make clear how strong the differential rotation of the mesh elements is. It also emphasizes the importance of allowing topology changes in the computational grid, which provide the required flexibility in order to preserve a high quality mesh over long times. Indeed, if the preservation of the connectivity had been imposed, the elements would have been quite distorted after rather short times. Finally, we would like to emphasize that generator trajectories are almost perfectly circular even for very long evolution times, which is quite an achievement!

Next, the Sedov explosion problem of Figure 5 and the triple point problem of Figure 6, which are benchmarks for moving mesh codes, demonstrate how we sharply fit strong shocks and show the clear advantages of using a Lagrangian method with respect to an Eulerian one. Furthermore, looking at Figure 7, one can appreciate the resolution given by our ALE-DG scheme, which captures, on a very coarse mesh, secondary structures of the Rayleigh -Taylor instabilities developed in stratified flows under the effects of gravity. We finally present two test cases for physical models other than the standard Euler equations, i.e. the rotor problem, benchmark of the magnetohydrodynamics equations for plasma physics, and an explosion problem in an elastic solid, showing an application of the so-called GPR unified model of continuum mechanics.

The accuracy of our results clearly show that the new combination of very high order schemes with regenerated meshes that allow topology changes may open a new prospective in Lagrangian type methods.

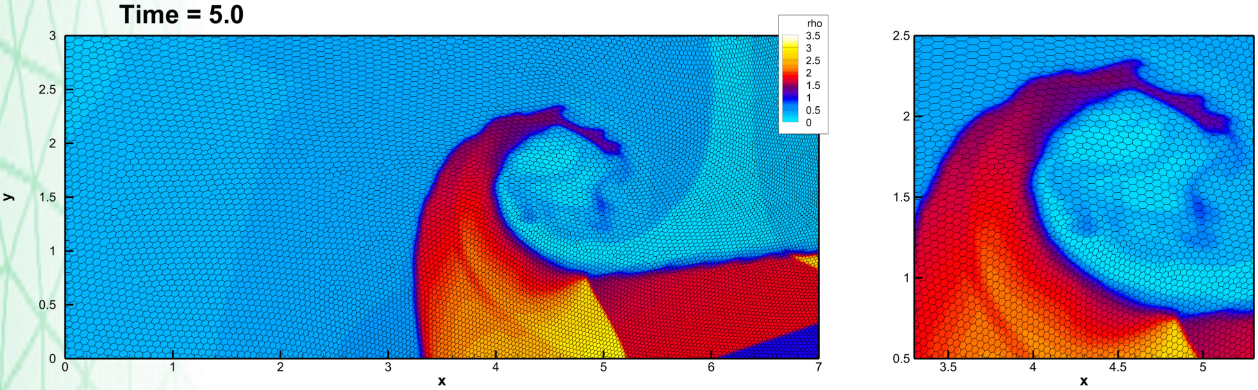

Figure 6: Triple point problem solved with our third order ALE-DG scheme.

Acknowledgments

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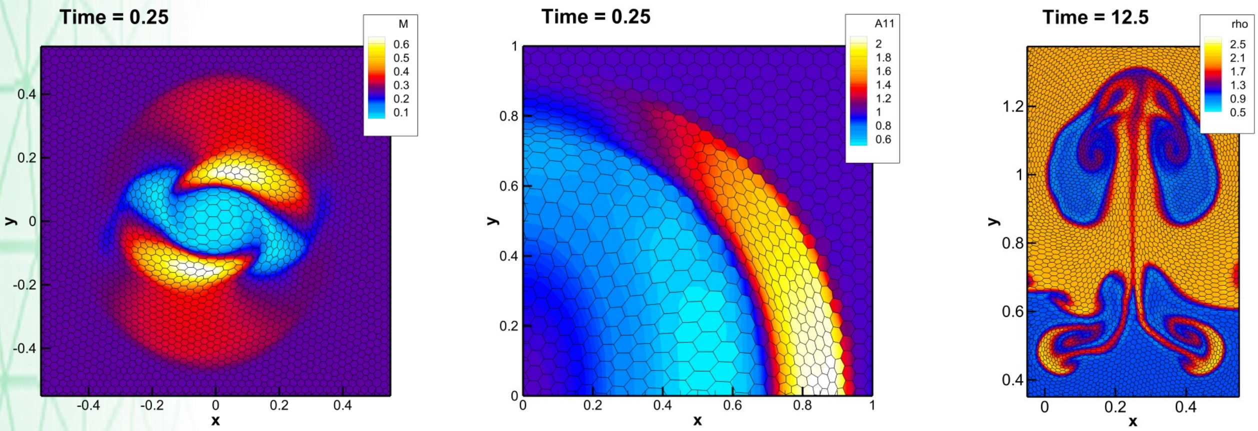

Figure 7: MHD rotor problem, explosion problem in an elastic solid, Rayleigh-Taylor instabilities.